

# Critical velocity for a toroidal Bose-Einstein condensate flowing through a barrier

F. Piazza<sup>1</sup>, L. A. Collins<sup>2</sup>, A. Smerzi<sup>3</sup>

<sup>1</sup> *Technische Universität München, Physik Department, James-Frank-Strae, 85748 Garching, Germany*

<sup>2</sup> *Theoretical Division, Mail Stop B214, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

<sup>3</sup> *INO-CNR and LENS, Largo Enrico Fermi 6, I-50125 Firenze, Italy*

We compare the theoretical prediction obtained with the Gross-Pitaevskii equation with recent experimental results [1] for the instability of the superfluid flow of a toroidal Bose-Einstein condensate in presence of a tunable weak link. The superflow with one unit of angular momentum becomes unstable at a critical strength of the link (implemented using a repulsive optical barrier), and decays through the mechanism of phase slippage performed by vortex-antivortex pairs. We find out the Gross-Pitaevskii prediction to be in agreement with the measured values of the critical barrier height. At the critical point the superfluid velocity in the vicinity of the obstacle is always of the order of the sound speed in that region,  $v_{\text{barr}} = c_1$ . The Feynman critical velocity  $v_f$  is instead much lower than the observed critical velocity, since we in general have  $v_f = \epsilon c_1$  where  $\epsilon$  is a small parameter of the model. Given the failure of the Feynman criterion, the question remains open whether the superfluid instability is energetical or dynamical.

Recent experiments performed at NIST [1, 2] and at the Cavendish Laboratory [3] have demonstrated the existence of persistent currents in a toroidally trapped Bose-Einstein condensate (BEC). Such a setup, apart from showing this hallmark manifestation of superfluidity [4], has also allowed, with the addition of a repulsive optical barrier, the construction of a closed-loop atom circuit with a weak link, as demonstrated in [1]. This circuit constitutes the basic building block for the realization of an ultracold atomic analog of superconducting/superfluid quantum interference devices such as the sensitive magnetic sensors already realized with superconductors (SQUIDS) [5] or rotational sensors with superfluid Helium (SHeQUIDS) [6]. Aside from the exciting technological applications, the toroidal setup is also the most suitable for the investigation of the critical velocity of a superfluid moving across an obstacle, since, compared to other setups [7–9] previously employed with harmonically trapped BECs, it does not require moving the obstacle since the condensate moves in the lab frame, and has no inhomogeneities along the direction of flow. The study of superfluid critical velocity and its decay mechanism is a long standing theoretical problem since the early experiments with superfluid Helium [10] and has not yet been fully understood.

In this manuscript, we show that the NIST ultracold atom circuit experiment [1] can be quantitatively well described by the Gross-Pitaevskii (GP) equation. In particular, the GP equation provides a very good prediction for the measured critical barrier height versus chemical potential for the superflow with one unit of angular momentum. This is a very important result since, as motivated above, the closed loop superfluid circuit is of deep foundational and technological interest. Firstly, the GP equation allows for a simple theoretical understanding of open questions regarding superfluid instability. Secondly, it can then be employed as a simple tool to model quantum interference devices, exploiting the versatility of ultracold atomic gases such as the fine tunability of the trapping potential, barrier potential, and the atom-atom

interactions.

At the critical point, beyond which vortex-antivortex pairs are nucleated, we observe that the superfluid velocity in the vicinity of the obstacle is always of the order of the local sound speed,  $v_{\text{barr}} = c_1$ , which is a typical feature of GP equation, at least close enough to the hydrodynamic regime [11–14]. We also observe that the Feynman critical velocity  $v_f$  [15] is instead much lower than the true critical velocity, namely  $v_f = \epsilon c_1$  where  $\epsilon$  is a small parameter of the model. This is another important message, since the measurement performed in [1], having no direct access to local quantities inside the barrier region, could not exclude the applicability of Feynman's prediction to the present situation.

The fact that the superfluid velocity in the vicinity of the obstacle is always of the order of the sound speed in that region might be taken, within a local density framework, as a signal that the Landau criterion is at work, with the corresponding excitations being thus the seeds from which vortices grow [16]. On the other hand, we have strong indications that the instability is of dynamical nature, since i) we do not have any defect inside the barrier region, and ii) the time scale of vortex nucleation is quite fast. We point out that the nature of the phase slip instability, energetic or dynamical, as well as its relation with the Feynman's criterion are not yet understood, and did not receive the appropriate attention in the past. We will conclude by summarizing the current level of understanding and emphasizing the open problems.

*Model.* We model the experiment performed at NIST [1] using the Gross-Pitaevskii (GP) equation in three dimensions subject to the external potential  $V(x, y, z) = (\omega_z^2/2)z^2 + (1/2)(\sqrt{x^2 + y^2} - R_0)^2$ , where the torus radius  $R_0 = 10$  and the vertical trap frequency  $\omega_z = 5$ . The results presented in the following are obtained by starting with an initial superfluid flow with angular momentum per atom  $\ell$  equal to one, then raising a  $x - y$  gaussian barrier centered at  $(x = R_0, y = 0)$  with widths  $\sigma_x = 7.55$  and  $\sigma_y = 2.15$ . Here the quantities are expressed in radial harmonic trap units: length  $d_x = 2 \mu\text{m}$ ,

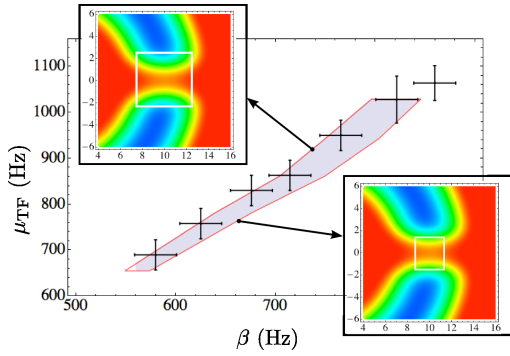


FIG. 1. (Color online) Chemical potential at the critical point for superfluid instability of the  $\ell = 1$  state as a function of the chemical potential decrease  $\beta$  inside the barrier region. Black dots with error bars: NIST measurement. Grey-shaded area: GP prediction for varying imaging resolution (see inset). Insets: column density in the vicinity of the barrier. Squares indicate the averaging region (accounting for finite imaging resolution) used to produce the curve delimiting the shaded area from above/below.  $x - y$  coordinates are expressed in harmonic radial trap units.

time  $\omega_x^{-1} = 1.45$  ms, and energy  $\hbar\omega_x = h \times 110$  Hz.

*Critical Barrier Height.* In Fig. 1, we show the comparison between the GP prediction for the critical point determining the superfluid instability and the experimental measurement performed at NIST. In order to allow for a comparison we have calculated the chemical potential, both global and local (inside the barrier region) with the same prescription [17] described in [1].

Given a total atom number  $N$ , we calculated the Thomas-Fermi chemical potential  $\mu_{\text{TF}}$  and subsequently obtained the chemical potential decrease  $\beta = \mu_{\text{TF}} - \mu_1$ , where  $\mu_1$  is the local chemical potential inside the barrier region. Instead of taking the actual local chemical potential by using  $n(x, y, z)$  directly from our numerical results, we inferred it following the NIST protocol, that is, taking the  $z$ -integrated column density  $\nu(x, y)$ , and assuming either a Gaussian or Thomas-Fermi profile along  $z$ , which provides a relation between  $\nu(x, y)$  and  $\mu_1$ . We did not actually use the column density  $\nu(x, y)$ , but rather its average  $\bar{\nu}$  over a square  $x - y$  region  $A$  centered about the barrier. The grey-shaded area in Fig. 1 is delimited by two curves, each corresponding to a single choice of  $A$  which we kept fixed for the whole chemical potential interval (see insets in Fig. 1).

For inferring the local chemical potential at the barrier, the use of the column density  $\nu$  averaged over a finite  $x - y$  region instead of simply  $\nu(R_0, 0)$  should yield a result closer to the experimental situation since this averaging accounts for a finite resolution of the experimental imaging system. We emphasize that the value of  $\beta$  strongly depends on the size of this region, that is, it strongly depends on the resolution of the imaging system, which we could not extract from [1]. In Fig. 1, a square area whose side length ranges from 5 to 10 micrometers provides good agreement. Also in [3], the role

of finite size resolution proves essential for the comparison between theory and experiment, and the width of the Gaussian point-spread function is about 3 micrometers, therefore fully compatible [18] with the size of the averaging region used in Fig. 1.

Our simulations thus show that the measurement of the critical barrier of [1] can be well described by the GP equation provided one accounts for the finite resolution of the imaging system used to extract the density inside the barrier. In [19], the same experimental data were compared with the prediction of both the GP equation and the truncated Wigner approximation. The model setup was a waveguide with periodic boundary condition, and the barrier assumed to be constant along the direction transverse to the flow. With the appropriate temperature, the Wigner approach agrees slightly better with the experimental data, but the error bars do not allow for a conclusive statement. However, the issue of the finite imaging resolution has not been considered.

*Criteria for instability.* In the Thomas-Fermi regime, where the system can be considered locally homogeneous inside the barrier region, we can rely on a precise criterion for superfluid flow instability in the GP equation. Indeed, it has been verified by simulating the time-dependent GP equation, both in toroidal and in waveguide geometries [14], that the critical point for superfluid instability in the Thomas-Fermi regime is determined by the condition that the fluid velocity at the Thomas-Fermi surface equals the effective sound speed. Actually, due to the density inhomogeneity transverse to the flow, the propagation speed of surface modes (excitations with larger momentum localized at the edge of the cloud [16]) is in fact lower than the sound speed [21], and is therefore first reached by the local fluid velocity. This also explains why the criterion of [14] involves the superfluid velocity at the Thomas-Fermi surface. However, the critical velocity for phonon instability and that for surface-mode instability differ significantly only very deep in the Thomas-Fermi regime [21] (the chemical potential must be larger than ten times the transverse trap frequency), which explains the fact that using the sound speed worked well in [14].

Fig. 2 depicts the typical situation at the critical point inside the barrier region showing the velocity profile (blue solid line) along a radial line ( $x = r$ ,  $y = 0$ ) crossing the middle of the barrier, where the velocity is largest, that is, where the critical condition is first reached. The green solid line is the value of the effective sound velocity corresponding to the above radial line. The effective sound velocity, which should be the relevant one in this geometry, is the speed of phonons propagating azimuthally in the  $x - y$  plane. It is calculated by considering the degrees of freedom along the  $z$  direction as frozen, and then averaging the local 2D sound velocity over the radial direction. If we assume a Gaussian profile along  $z$ , we get  $c_{\text{eff}}^{(2D)} = \sqrt{(2/3)g/(m\sqrt{2\pi})\nu(R_0, 0)/d_z}$ , where  $g = 4\pi a_s \hbar^2/m$ , and  $\nu(R_0, 0)$  is the column density at the

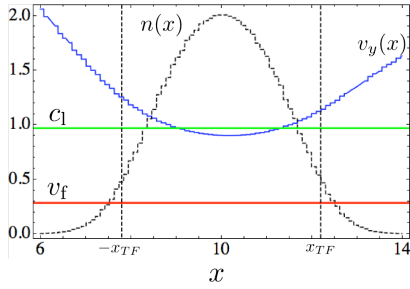


FIG. 2. (Color online) Fluid velocity at the critical point for superfluid instability of the  $\ell = 1$  state. Blue solid line: azimuthal component of the local fluid velocity along a radial line crossing the middle of the barrier region. Both the radial coordinate  $x$  and the velocity  $v(x)$  are expressed in harmonic radial trap units. Black solid line: density (arbitrary units) along the same radial line with vertical dashed black lines indicating the position of the Thomas-Fermi surface. Green solid line: sound velocity inside the barrier region  $c_1 = c_{\text{eff}}^{(2D)}$  for phonons propagating azimuthally in the  $x - y$  plane. Red solid line: Feynman's prediction with  $D = 5d_x$  and  $a = \xi_1$ . The chemical potential is  $\mu_{TF} = \hbar \times 1030\text{Hz}$ .

center of the barrier.

As we see from Fig. 2, the local criterion described above is not very well verified since, due to the high barrier, the density is strongly depleted at the critical point, and therefore we are outside of the Thomas-Fermi regime. Nonetheless, we can see that the local fluid velocity inside the barrier region  $v_{\text{barr}}$  is still of the order of the effective sound speed  $c_1$ . This is a general feature of the superfluid instability point at the GP level, provided that the condensate in the bulk region (outside the barrier) be not too far from the Thomas-Fermi regime.

The red solid line in Fig. 2 corresponds instead to the Feynman prediction for the critical velocity  $v_f = \hbar/(mD) \ln(D/a)$ , where  $D$  is the annulus width and  $a$  is the vortex core size. For the latter we took the healing length  $a = \xi_1 = (1/\sqrt{2})\hbar/mc_1$  corresponding to the effective sound speed  $c_1$  inside the barrier region. While the width of the annulus is  $D \simeq 8d_x$  outside the barrier region, it is reduced inside:  $D \simeq 5d_x$  (we took the Thomas-Fermi radial width of the torus inside the barrier region). We see that the Feynman prediction  $v_f$  is significantly lower than fluid velocity inside the barrier region and is even lower than the fluid velocity outside the barrier region  $v_{\text{bulk}} \simeq 0.7d_x\omega_x$ . In general, independently of the geometry, and at least for angular momenta per atom  $\ell < 10$ , we always found with GP that the Feynman prediction is typically much lower than the fluid velocity in the barrier region. The generality of this fact becomes apparent if we rewrite the Feynman velocity as

$$v_f = c_1 \sqrt{2} \frac{\xi_1}{D} \ln\left(\frac{D}{\xi_1}\right) = \epsilon c_1. \quad (1)$$

As stated before, the superfluid instability point at the GP level corresponds in general to a situation where the local fluid velocity inside the barrier region  $v_{\text{barr}}$  is of

the order of the effective sound speed  $c_1$ . Now, since the expression for the Feynman velocity given above is only valid when the channel width (in our case the annulus width) is much larger than the vortex core size, i.e.  $\tilde{D} = D/\xi_1 \gg 1$ , we have  $v_f = \epsilon c_1$ , where  $\epsilon = \sqrt{2} \ln(\tilde{D})/\tilde{D} \ll 1$  is a small parameter in the theory. Our observations are consistent with the experiment performed at the Cavendish Laboratory [3], where the Feynman prediction is typically three times smaller than the observed critical velocity. Moreover, Eq. (1) indicates that the Feynman prediction is not simply a very rough estimate of the true GP critical velocity but is instead intrinsically different, since it cannot be applicable to instabilities whose critical point corresponds to a superfluid velocity being of the order of the sound speed.

On the other hand, the measurements performed at NIST [1] seemed to suggest that the Feynman prediction may work well. This can be due to the presence of a macroscopic barrier that strongly depletes the density, making the experimental determination of the local quantities inside the barrier much more difficult. Indeed, we verify that the estimation of the local chemical potential  $\mu_l$  using the column density overestimates the local chemical potential (the worse the imaging resolution the larger is the overestimation), which in turn underestimates the healing length and therefore overestimates  $v_f$ . Secondly, the local fluid velocity inside the barrier is underestimated by solving the equivalent 1D flow problem obtained by integrating radially the measured column density. We can therefore conclude that the estimation of the local quantities inside the barrier region in [1] is too crude to allow us to use the measured data, e.g. the local superfluid velocity, to discriminate between different theoretical predictions.

In Fig. 3, an example of the superfluid decay dynamics is shown. Two different views, referring to the same instant after the barrier has reached the critical height, depict two straight vortex lines with opposite winding numbers moving toward the center of the annulus. The two lines eventually meet close to the outer edge of the annulus and annihilate, thereby producing a full phase slip which brings the angular momentum to zero. One line comes from the low density region at the torus center while the other moves inwards from the system boundary. The entire process is very fast, taking about 0.5 ms. The inner line enters the bulk first since the instability is reached first at the inner edge of the annulus due to the velocity asymmetry [23]. This asymmetry characteristic of a torus has been observed to influence the Landau criterion for a BEC in a torus without barrier [20], where the surface modes become unstable first at the inner edge. Since the decay dynamics involve straight vortex lines, it is reasonable to ask ourselves what prediction we would get if we modify the above Feynman velocity, related to vortex rings, to the case of nucleation of vortex lines. As discussed in [22], the Feynman critical velocity for the nucleation of a vortex line crossing the channel differs only by a prefactor  $2/\pi$  from the critical velocity for ring nu-

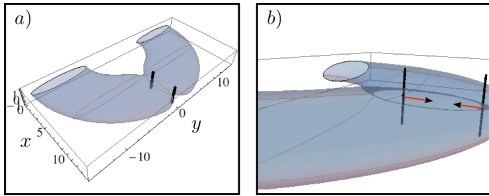


FIG. 3. (Color online) Superfluid decay dynamics of the  $\ell = 1$  state. The parameters are the same as in Fig. 2. a) and b) represent two different viewpoints of the same moment of the dynamics. Two straight vortex lines move toward each other.

cleation.

*The nature of the instability.* At this point some questions should arise: 1) why is the Feynman criterion incompatible with the GP, and what is its relation, if any exists, with the local criterion for the instability of surface modes? 2) Is the true instability energetic or dynamical? In the following we will not answer these questions, but rather argue that they are well posed and that no clear answers have yet been given despite their fundamental relevance for the understanding of superfluidity.

1) The physical reason underlying the failure of the Feynman prediction, verified here with the GP equation and experimentally in [3], is not understood. It can be argued that this is due to the inhomogeneity along the direction transverse to the flow, which however does not justify the intrinsic incompatibility (see Eq. (1)) with GP instabilities where the critical velocity is of the order of the sound speed. On the other hand, one might suppose that the Feynman criterion could work well when additional defects are present inside the barrier region. In general, a useful insight might be provided by relating the Feynman criterion, based on vortices, to the instability of surface modes [24].

2) The first observation is that we have a quantitatively viable criterion for predicting the critical velocity. As discussed above, in the Thomas-Fermi regime, we can precisely define the critical condition as the superfluid velocity at the Thomas-Fermi surface of the cloud being equal to the speed of propagation of surface modes inside the barrier region. This criterion has also been previously theoretically verified to a reasonable degree for the case of a BEC flowing past an obstacle in [11–14], and for a rotating elliptical trap [26]. Remarkably, it has been even experimentally verified at the Cavendish Laboratory [3] where the instability is attributed to the roughness of the trap creating a region where the flow is constricted. In general, this criterion takes the excitation spectra calculated in the system without anisotropy along the flow direction, for instance in a transversally inhomogeneous waveguide [16, 21, 27, 28], a rotationally symmetric trap [29–32], or a torus without barrier [20], and applies the Landau criterion to the anisotropic case locally inside the region where the superfluid velocity is higher, assuming the Thomas-Fermi approximation to hold along the flow

direction. Therefore, since the fluid is locally homogeneous (inside the barrier the system does not feel the edges of the latter), it has been suggested in the literature that the instability is energetic.

However, the second observation is that time-dependent GP simulations give strong indications that the instability is dynamical, since: i) we do not have any defects inside the barrier region (or in general the “high velocity region”) which would be required in order to trigger the energetic instability [33], and ii) the nucleation dynamics is generally very fast, which means that excitations grow very quickly which is rather typical for dynamical instabilities, whose onset can indeed be observed also in absence of defects.

We note that this kind of apparent ambiguity appears also in the studies of the critical angular velocity for a BEC in a rotating harmonic trap. In particular, using a linear stability analysis in presence of anisotropy, the surface modes have been shown to become dynamically unstable [34], and the critical angular frequency to agree with the real-time propagation of the GP equation [35]. At the same time however, the real-time GP results in the anisotropic trap have been observed [26] to be also well predicted by the “local Landau criterion” described above. To our knowledge, a conclusive statement about the relation between the local Landau criterion and dynamical instability is still missing in this context.

It seems in general that the critical velocity for the onset of vortex nucleation might correspond to a dynamical instability, which, in the Thomas-Fermi regime, is, remarkably, signaled by the Landau criterion applied inside the locally homogeneous constriction region. An example which could sustain this thesis is given by a one-dimensional condensate flow crossing the speed of sound twice [36]. In this setup, a linear stability analysis predicts a dynamical instability as soon as the central region becomes supersonic, that is, the Landau criterion is fulfilled. The correspondence between the local Landau criterion and the global dynamical instability seems crucial to understand the mechanism underlying vortex nucleation and deserves further study.

*Conclusions.* We have shown that the superfluid decay experiments performed with a toroidal BEC flowing past a repulsive barrier can be well described by the GP equation, provided the finite resolution of the imaging system is taken into account. We have also shown that for this setup the Feynman criterion inevitably fails to predict the GP critical velocity, since the latter is of the order of the sound speed inside the constriction region. The relation between the Feynman criterion and the GP phase slip instability, as well as a conclusive statement about the nature of the latter did not find a satisfactory understanding so far, and deserve further study.

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